

# Bistable and dynamic states of parametrically excited ultrasound in a fluid-filled cavity.

Isabel Pérez-Arjona, Víctor J. Sánchez-Morcillo and Víctor Espinosa

*Instituto de Investigación para la Gestión Integrada de las Zonas Costeras,*

*Universidad Politécnica de Valencia, Crta. Natzaret-Oliva s/n,*

*46730 Grau de Gandia (Valencia), Spain*

## Abstract

In this paper we have considered the problem of parametric sound generation in an acoustic resonator filled with a fluid, taking explicitly into account the influence of the nonlinearly generated second harmonic. A simple model is presented, and its stationary solutions obtained. The main feature of these solutions is the appearance of bistable states of the fundamental field resulting from the coupling to the second harmonic. An experimental setup was designed to check the predictions of the theory. The results are consistent with the predicted values for the mode amplitudes and parametric thresholds. At higher driving values a self-modulation of the amplitudes is observed. We identify this phenomenon with a secondary instability previously reported in the frame of the theoretical model.

## I. INTRODUCTION

Ultrasonic resonators are devices that confine the acoustic fields in a finite region of the space. When driven by an external energy source (e.g. vibrations of one of their boundaries), the reflection and resonance conditions imposed by the boundaries provide the conditions to obtain high amplitude fields, favoring the development of nonlinear effects, as the emergence of frequencies different from that of the driving.

The knowledge of the field evolution in resonators in nonlinear regime is important both in its fundamental and applied aspects. One of the simplest configurations consists in two plane and parallel walls (an acoustic interferometer) where nearly one-dimensional standing waves are formed along the cavity axis. The system was the basis, forty years ago, of the first experimental observation of acoustic parametric excitation in a fluid [1, 2]. The phenomenon, first observed by Faraday [4] and later described by Lord Rayleigh, consists in the emergence of oscillation modes at frequencies smaller than that of the driving, when a parameter of the system is varied periodically in time. The phenomenon is universal, and has been demonstrated in a variety of physical systems [5]. In the case of the acoustical interferometer, the length of the cavity (and thus the eigenfrequencies of the normal modes) is the time-dependent parameter, and the parametric excitation is achieved when the input energy is high enough to overcome the dissipative losses. The parametric fields usually appear as doublets, whose frequencies  $f_1$  and  $f_2$  add to match the driving frequency  $f_0$ , i.e.  $f_0 = f_1 + f_2$ , although in some circumstances the half-frequency mode is observed.

A theoretical description, based on the Mathieu equation, has been successfully applied to the description of these processes in an acoustical interferometer, allowing to predict the subharmonic spectrum and its excitation threshold [5, 6]. This approach has however a restricted validity, since it can not describe the further evolution of the parametric fields above the threshold and, on the other hand, it ignores the unavoidable effects of the higher harmonics of the driving, which are the main signature of non- or weakly-dispersive acoustic systems.

The spectrum of higher harmonics can be controlled using additional dispersion mechanisms, such as bubbly fluids or walls with selective (frequency dependent) absorption [7, 11]. Ostrovsky et al [10] designed a waveguide cavity where dispersion is introduced by the lateral boundaries, and the higher harmonics could be completely inhibited. As a theoretical

framework, a set of evolution equations for the trial of interacting modes was considered, resulting in an excellent agreement with the measured amplitudes above threshold.

This theory has been also applied to the interferometer case [7, 12, 13, 14], but the agreement with the experiment [7] was mainly qualitative. The discrepancies can be interpreted in terms of the influence of the first higher harmonics (those with higher amplitudes) on the parametric process, introducing additional features that can not be captured by the fully dispersive model.

In this work we present a theoretical description of parametric sound generation in a fluid-filled interferometer, extending the previous models to include the coupling with the second harmonic. The stationary solutions both above and below the threshold are obtained and discussed. The solutions are compared with experimentally measured values, with an excellent agreement. The novel effects induced by the second harmonic (e.g. bistability or hysteresis) are discussed. Finally, we also report the existence of low frequency self-oscillations, occurring at pump values beyond the parametric instability threshold. This phenomenon is discussed in terms of secondary instabilities of the proposed model.

## II. THEORY

The acoustical interferometer considered in this paper is composed by two parallel and solid walls, with thicknesses  $D$ , located at a distance  $L$  from each other, containing a fluid medium inside. Each medium involved in the model is acoustically characterized by its density  $\rho$ , bulk modulus  $\kappa$ , and sound velocity  $c$ , related as  $c^2 = \frac{\kappa}{\rho}$ . The resonance modes (eigenfrequencies) of the resonator depend on these parameters, defining the acoustical impedances  $z = \rho c$  of each section. In the ideal (loseless) case, corresponding to an infinite impedance of the walls, the resonance modes are equidistant and obey the equation

$$\tan(k_f L) = 0 \quad (1)$$

where  $k_f = 2\pi f/c_f$ . However, in a real system the impedances have finite values, and the spectrum of the resonator is no longer equidistant, but distributed according to the transcendental equation

$$\tan(k_f L) = \frac{2\mathcal{R} \tan(k_w H)}{\mathcal{R}^2 \tan^2(k_w H) - 1}, \quad (2)$$

where  $k_w = 2\pi f/c_w$  and  $\mathcal{R}$  is the quotient between wall and fluid impedances.

Now consider the driven system, assuming that one of the walls vibrate with frequency  $f_0$ . Then, above the parametric generation threshold the spectrum inside the resonator can be decomposed in two sets: the subharmonics resulting from the parametric instability and the higher harmonics of the driving,  $nf_0$ , with  $n$  an integer number, resulting from the weak dispersion in the system. The amplitude of any of these harmonics decrease with the detuning, defined as the difference between the corresponding field frequency  $\omega_n$  and the frequency of the closest cavity mode,  $\omega_n^c$ , *i.e.*  $\delta_n = \omega_n - \omega_n^c$ . The energy transfer into a harmonic is then more effective under resonance conditions. In an ideal cavity, a resonant driving implies that all higher harmonics are also resonant with a cavity mode, enhancing the cascade of energy from the driving into many higher frequency components. However, an initial detuning  $\delta_0$  implies that the detuning of the higher order modes increases linearly with the frequency (Fig. 1(a)). As a consequence, the amplitude of these modes is reduced with respect to the resonant case. This situation can be more dramatic in a real system, where the mode distribution is non-equidistant and the detuning of the second and higher harmonics can be tuned to be much larger than that of the driving and the subharmonics, as shown in (Fig. 1 (b)).

The above arguments suggest that a theoretical approach to parametric sound generation must take into account the effect of the second harmonic, whose amplitude is not negligible in general, but one can neglect the influence of the third and higher harmonics, assuming that they are sufficiently out of resonance. To simplify the analysis, we consider that the parametric generation is degenerated, *i.e.*, the half harmonic of the driving is excited.

Under these assumptions, and following the technique described in detail in [13] for the fully dispersive case (neglecting higher harmonics), the following system of equations for the evolution of the slowly-varying amplitudes for each mode is obtained:

$$\begin{aligned}\frac{dP_0}{dt} &= -(\gamma_0 + i\delta_0)P_0 - i\beta(P_1^2 + P_2P_0^*) + \frac{c}{L}P_{in}, \\ \frac{dP_1}{dt} &= -(\gamma_1 + i\delta_1)P_1 - i\frac{\beta}{2}P_0P_1^*, \\ \frac{dP_2}{dt} &= -(\gamma_2 + i\delta_2)P_2 - i\beta P_0^2,\end{aligned}\tag{3}$$

corresponding to the fundamental (driving), subharmonic and second harmonic respectively. Other parameters are:  $P_{in}$  the driving amplitude (pump),  $\gamma_i$  the decay rates of each mode in the cavity,  $\beta = \frac{\omega_0}{4\rho c^2}(1 + B/A)$  is related to the nonlinearity parameter  $B/A$  of the fluid and  $\delta_i = \omega_i - \omega_i^c$  are the detunings.

The dynamical system given by Eqs.(3) can be reduced to a simpler, dimensionless form, defining the new variables  $A_0 = i\frac{2\gamma_1}{\beta}p_0$ ,  $A_1 = i\frac{\sqrt{2\gamma_0\gamma_1}}{\beta}p_1$ ,  $A_2 = i\frac{\gamma_0}{\beta}p_2$ ,  $E = i\frac{2L\gamma_0\gamma_1}{\beta}p_{in}$ , and the parameters  $\gamma = \frac{\gamma_1^2}{\gamma_0\gamma_2}$  and  $\Delta = \delta_i/\gamma_i$ . With these changes we obtain

$$\begin{aligned}\gamma_0^{-1}\frac{dA_0}{dt} &= -(1+i\Delta_0)A_0 + E - A_1^2 - A_2A_0^*, \\ \gamma_1^{-1}\frac{dA_1}{dt} &= -(1+i\Delta_1)A_1 + A_0A_1^*, \\ \gamma_2^{-1}\frac{dA_2}{dt} &= -(1+i\Delta_2)A_2 + i\gamma A_0^2.\end{aligned}\tag{4}$$

Equations (4) admit two different stationary solutions. When the pump amplitude  $E$  is below the parametric threshold, one can set  $A_1 = 0$ . Neglecting the temporal derivatives in Eqs.(4) we obtain

$$\begin{aligned}E^2 &= |A_0|^2 \left( 1 + \Delta_0^2 + \frac{\gamma |A_0|^2 (\gamma |A_0|^2 + 2(1 - \Delta_0\Delta_2))}{1 + \Delta_2^2} \right)^{1/2} \\ |A_2|^2 &= \frac{\gamma^2}{1 + \Delta_2^2} |A_0|^4\end{aligned}\tag{5}$$

where we have defined  $I_0 = |A_0|^2$  and  $I_2 = |A_2|^2$ .

The solution given in Eq.(5) reflects the fact that the amplitudes of both the fundamental and the second harmonic grow with the pump amplitude, as expected. The most interesting feature is however the emergence of multivalued solutions, *i.e.* the system shows a bistable or hysteretic behavior, when some conditions on the parameters are fulfilled. The condition for multivaluedness is found by imposing the existence of an inflection point in the S-shaped curve given by Eqs.(5). This occurs when  $\frac{d^2E^2}{dI_0} = 0$ , *i.e.* at

$$|A_0|^2 = -\frac{2(1 - \Delta_0\Delta_2)}{3\gamma}.\tag{6}$$

Note that the existence of bistable solutions requires that  $\Delta_0\Delta_2 > 1$ , *i.e.* both detunings must have the same sign and exceed a certain critical value. Figure 2 shows the monostable (a), critical (b) and bistable (c) cases of the fundamental mode, for different sets of parameters.

At a pump value given by

$$E_{th} = \sqrt{(1 + \Delta_1^2) [(1 + \tilde{\gamma})^2 + (\Delta_0 - \tilde{\gamma}\Delta_2)^2]}\tag{7}$$

the solution given by Eq.(5) becomes unstable, denoting the threshold of the parametric instability. The new solution corresponds to a non-zero value of the subharmonic amplitude, given by

$$|A_1|^2 = (-1 + \Delta_0\Delta_1) - \tilde{\gamma}(1 + \Delta_1\Delta_2) \pm \sqrt{P^2 - [\Delta_0 + \Delta_1 + \tilde{\gamma}(\Delta_1 - \Delta_2)]^2} \quad (8)$$

where we have defined  $\tilde{\gamma} = \gamma \frac{1+\Delta_1^2}{1+\Delta_2^2}$ .

Above the instability threshold, the fundamental mode saturates to a constant value

$$|A_0|^2 = 1 + \Delta_1^2, \quad (9)$$

and  $A_2$  is given by Eq.(5).

Note that in the limiting cases of large second harmonic detuning  $\Delta_2$  or large losses  $\gamma_2$  (in both cases  $\tilde{\gamma}$  tends to zero), the solutions (5)-8 reduce to those obtained in previous works (see e.g. [13]) for the fully dispersive cavity. Physically a mode with large detuning or losses is strongly damped in the cavity, so in practice it can be neglected from the very beginning.

The parametric instability can be either supercritical or subcritical, resulting in monostable or bistable subharmonic respectively. This fact, different to the bistability of the fundamental mode, is also present in the fully dispersive cavity, as shown in [10, 12].

Finally we note that, in order to observe the bistable regime of the fundamental mode, it must occur with precedence to the parametric instability. Combining Eqs.(6) and (9), we find that the fundamental mode presents bistability whenever

$$0 < \frac{2}{3\gamma} (\Delta_0\Delta_2 - 1) - \Delta_1^2 < 1.$$

### III. EXPERIMENT

#### A. Description

The resonator consists in two piezoceramic discs ( $\rho = 7.70 \times 10^3 \text{kg/m}^3$ ,  $c = \text{m/s}$ ) with radius 1,5 cm, and thickness 1mm and 2 mm (corresponding to resonance frequencies around 2 MHz and 1MHz, respectively), mounted in a Plexiglas tank containing distilled and degassed water ( $\rho = 1.00 \times 10^3 \text{kg/m}^3$ ,  $c = 1480 \text{m/s}$  at  $T = 20^\circ\text{C}$ ). Both sides are located at a variable distance  $L$ , and its parallelism can be carefully adjusted to get a high-Q interferometer. One of the piezoceramics, that with resonance frequency around 2 MHz is

driven by the signal provided by a function generator (Agilent 33220) and a broadband RF power amplifier ENI 240L. The experimental setup is completed by a needle hydrophone (TNUA200 NTR Systems) to measure the intracavity pressure field. The Fourier transform of the acquired signal allows to determine the spectral content of the resonator, and from it to quantify the amplitude of any frequency component.

In this way, by changing the amplitude of the driving source, we are able to follow the bifurcation diagram of the resonator for a given set of parameters. Although the pump value and the decay rates  $\gamma_i$  can be unambiguously determined (the latter by measuring the line width of the cavity modes), the detuning parameters  $\delta_i$  are difficult to control, and in general vary from one set of measurements to the next. The reason is the dependence of the cavity resonances with the temperature of the medium, which changes with time due to the external sources (heating of the driving transducer or ambient temperature variations). However, we have monitored the temperature variations to ensure that the detunings did not changed appreciably withing one set of measurements, so we could apply the theory of the previous section to the description of this problem.

## B. Bifurcation diagrams

In the series of experiments we have obtained different bifurcation diagrams. They differ in the spectral content (frequency of the subharmonic pairs, weight of the higher harmonics) corresponding to different sets of detunings, but also in the behavior of the individual amplitudes. We illustrate two of such cases in Figs. 3 and 4.

Figure 3 was obtained for a driving frequency  $f_0 = 1.991$  MHz and  $L = 3.5$  cm. In this case a nearly-degenerated subharmonic pair, with frequencies  $f_1 = 1.018$  MHz and  $f_2 = 0.974$  MHz, is excited. As shown in Fig. 3(a), corresponding to a driving amplitude of 25 V, the pair is highly asymmetric, the amplitude of one of the components,  $f_2$ , being negligible with respect to  $f_1$ . The experimental bifurcation diagram is shown in Fig.3(b) with symbols. The first evident feature of the diagram is the non-linear growth of the fundamental mode amplitude as the driving amplitude  $V_{in}$  is increased, denoting a non-trivial influence of the second harmonic. At  $V_{in} = 24$  V, the threshold of parametric generation is reached, and the amplitude of the subharmonics begins to grow. Note that in Fig.3(b) this amplitude is magnified five times for a better clarity.

The solid lines represent the analytical stationary solutions given by Eqs.(5), (8) and (9). The theoretical amplitude and pump values have been scaled by respective numerical factors taking into account the efficiency of the transducer, the sensitivity of the hydrophone, and the additional normalizations leading to Eqs.(4). We used the medium parameters  $\rho = 1.00 \times 10^3$  kg/m<sup>3</sup>,  $c = 1480$  m/s and  $\sigma = 0.875$ . The decay rates were obtained by measuring the line width of the modes, resulting  $\gamma_j \sim 5000$  s<sup>-1</sup>. Finally, the detunings have been chosen to get the best fit to the experimental data. In Fig.3(b) we used  $\Delta_0 = -2.35$ ,  $\Delta_1 = 3.25$  and  $\Delta_2 = -4$ .

For a slightly different driving frequency  $f_0 = 1.879$  MHz, we observed a different scenario, shown in Fig. 4. In this case a pair frequencies  $f_1 = 1.242$  MHz and  $f_2 = 0.638$  MHz is excited above threshold. The pair is also clearly asymmetric in the amplitudes, as shown in Fig.4(a) corresponding to a driving amplitude of 8 V. Different from the previous case a linear growth of the fundamental mode below threshold is observed in Fig.4(b), saturating at a constant value above threshold (7 V in this case). As discussed in the previous section, this situation is typical of a highly detuned second harmonic, where the fully dispersive model represents a good approach. The theoretical bifurcation diagrams in Fig.4(b) were obtained for the set of detunings  $\Delta_0 = 1.35$ ,  $\Delta_1 = 0.6$  and  $\Delta_2 = 10$ , and show a good agreement with the measured values.

### C. Self-modulation

Further increasing the pump above the threshold the field components often develop low frequency side bands, as shown in Fig.5. The parameters are those corresponding to Fig.4, and the pump parameter is nearly twice the parametric threshold value. In the temporal domain, the presence of the side bands implies a low frequency modulation of the field amplitudes. This phenomenon, also demonstrated in solids [15] is usually called self-modulation since the source amplitude is kept constant. Self-modulation can result from different mechanisms, as discussed in [16].

In [14] we considered the stability of the solutions of Eqs.(8) and (9), identifying the appearance of the self-modulation with the Hopf bifurcation that the amplitudes undergo under certain conditions. The frequency of the slow modulation predicted by the theory was of the order of several kilohertz, for typical operation conditions. This is in agreement with



the observed side band frequency,  $\Delta f = 34$  kHz, in Fig. 5. A careful analysis of the spectrum allows to identify these new frequencies as the result of the mixing (sum and difference,  $f_i \pm f_j$ ) between the fundamental and the subharmonics components, with the second harmonics  $2f_i$ , owing to the quadratic nonlinearity of the medium. Some of these frequencies are correspondingly labeled in Fig. 5. The experiment also shows that the appearance of the low side band frequencies occurs once the pump reaches a critical value. This behaviour of the spectrum is characteristic of a secondary instability of the Hopf type, in agreement with the analysis performed in [14].

#### IV. CONCLUSIONS

The models proposed previously to describe the parametric sound generation in acoustic resonators can not account for a number of effects observed in the experiment. The first reasonable approach to this problem is to consider the role played by the first higher harmonics, whose magnitude is strongly dependent of the distance to the cavity resonances. The proposed model, which is a simple extension of previous (dispersive) models, considers the coupling of the fundamental field to its second harmonic. We demonstrate that the bifurcation diagrams predicted by the theory and observed in the experiment can be qualitatively different under different parameters, and are very sensible to the experimental conditions (mainly the driving frequency and the temperature, both affecting the detuning values). Such model also predicts the emergence of secondary instabilities leading to time self-modulated solutions, in agreement with the experimental observation at drivings beyond the threshold. We expect that, at even higher drivings, the dynamics of the system be chaotic, as the theory predicts. Experimental work in the direction is in progress.

## Acknowledgments

The work has been financially supported by the MEC of the Spanish Government, under the project FIS2005-0793-C03-02.

---

- [1] A. Korpel and R. Adler, Parametric phenomena observed on ultrasonic waves in water, *Appl. Phys. Lett.* **7**, 106 (1965).
- [2] L. Adler and M.A. Breazeale, Excitation of subharmonics in a resonant ultrasonic wave system, *Naturwiss.* **55**, 385 (1968).
- [3] L. Adler and M.A. Breazeale, Generation of fractional harmonics in a resonant untrasonic wave system, *J. Acoust. Soc. Am.* **48**, 1077 (1970).
- [4] M. Faraday, *Phil. Trans. Roy. Soc. (London)* **121**, 299 (1859)
- [5] L. Adler and M. A. Breazeale, Parametric phenomena in physics, *Am. J. Phys.* **39**, 1522-1527 (1971)
- [6] A. Teklu, M.S. McPherson, M A. Breazeale, R.D. Hasse, N.F Declercq, Measurement of the frequency dependence of the ultrasonic parametric threshold amplitude for a fluid-filled cavity, *J Acoust Soc Am.* **120**, 657 (2006)
- [7] N.Yen, Experimental investigation of subharmonic generation in an acoustic interferometer, *J. Acoust. Soc. Am.* **57**, 1357 (1975).
- [8] L.A. Ostrovsky and I.A. Soustova, Theory of parametric sound generators, *Sov. Phys. Acoust.* **22**, 416-419 (1976).
- [9] F.V. Bunkin, Yu. A. Kravtsov and G.A. Lyskhov, Acoustic analogues of nonlinear-optics phenomena, *Sov. Phys. Usp.* **29**, 607-619 (1986).
- [10] L.A. Ostrovsky, I.A. Soustova and A.M. Sutin, Nonlinear and parametric phenomena in dispersive acoustic systems, *Acustica* **39**, 298-306 (1978).
- [11] L.K. Zarembo and O.Y. Serdobolinskaya, Parametric amplification and generation of acoustic waves, *Sov. Phys. Acoust.* **20**, 440-443 (1975).
- [12] V.J. Sánchez-Morcillo, Spontaneous pattern formation in an acoustical resonator, *J Acoust Soc Am.* **115**, 111 (2004)
- [13] I. Pérez-Arjona and V.J. Sánchez-Morcillo, Pattern formation in parametric sound generation,

- Phys. Rev. E **72**, 066202(2005)
- [14] V.J. Sánchez-Morcillo, V. Espinosa, J. Redondo and J. Alba, Nonlinear dynamics and chaos in parametric sound generation, *Acta Acustica United with Acustica* **92**, 210-216 (2006)
- [15] I. Solodov, J. Wackerl, K. Pfeiderer, and G. Busse, Nonlinear self-modulation and subharmonic acoustic spectroscopy for damage detection and location, *Appl. Phys. Lett.* **84**, 26 (2004)
- [16] L. Fillinger, V. Zaitsev, V. Gusev and B. Castagnede, Wave self-modulation in an acoustic resonator due to self-induced transparency, *Europhys. Lett.*, **76** (2), 229-235 (2006)

*a. Figure Captions* Figure 1. Schematic representation of the field and cavity spectra, in the cases of (a) ideal (loseless) cavity, (b) a real cavity with finite impedances.

Figure 2. Development of bistability of the fundamental mode as the second harmonic detuning is varied. Parameters are  $\gamma = 1, \Delta_0 = -2$ , and  $\Delta_2 = 1$  (a),  $\Delta_2 = 0$  (b), and  $\Delta_2 = -1$  (c).

Figure 3. Experimental bifurcation diagram (symbols) obtained for a driving frequency  $f_0 = 1.991$  MHz. Solid and dashed lines correspond to the fundamental and subharmonic amplitudes, as obtained from Eqs. (5) and (8) for  $\Delta_0 = -2.35$ ,  $\Delta_1 = 3.25$  and  $\Delta_2 = -4$ .

Figure 4. Experimental bifurcation diagram (symbols) obtained for a driving frequency  $f_0 = 1.879$  MHz. Solid and dashed lines correspond to the fundamental and subharmonic amplitudes, as obtained from Eqs. (5) and (8) for  $\Delta_0 = 1.35$ ,  $\Delta_1 = 0.6$  and  $\Delta_2 = 10$ .

Figure 5. Development of low-frequency side bands for the experimental conditions in Fig. 4, for a driving amplitude  $V_{in} = 17V$ .











